A Mosaic of Multivariate Calculus (CDT-27)

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Abstract

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"A passion for calculus can unlock new worlds."

Isaac Newton.

1 Introduction

Developed by Isaac Newton and Gottfried W. Leibniz, integro-differential calculus (e.g. [1, 2, 3, 4]) stands as one of the most fundamental and important resources physical science and technology, specially in the physical sciences and scientific modeling (e.g. [5]). Basically, multivariate calculus studies the properties of relatively smooth mathematical structures and maps, such curves and surfaces along, a wide range of scales, going from infinitesimal intervals to infinitely ample integrals. Its applications are virtually unlimited, so that a good command of its basic principles and methods can provide considerable resources for both theoretical and applied researchers.

While univariate calculus, dealing with functions on a one-dimensional domain, is already rich and ample, its multivariate counterpart is even ampler and applicable from both theoretic and applied perspectives — after all we live in a multivariate world. Multivariate calculus is also closely related to other mathematical areas, such as analytic geometry, linear algebra, and differential geometry.

In multivariate calculus, unlike in one-variable calculus, the considered curves and surfaces are defined not necessarily on an one-dimensional domain, but can extend over many dimensions. Of special interest are scalar fields, which associates a scalar value to every point of a respective multidimensional domain. Scalar fields, the direct counterpart of functions in one-dimensional calculus, can be used to build composed maps such as multivariate functions and vector fields.

In a similar manner to the relationship between univariate and multivariate statistical modeling (e.g. [6, 7]), multivariate calculus focuses on many concepts and methods related to the *interaction* between variables and maps. As these multivariate approaches are based on univariate concepts, it is desirable for the reader to have some experience with univariate calculus (e.g. [3, 4]).

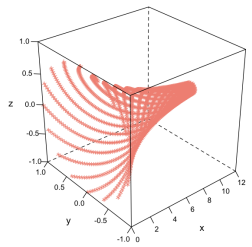

Figure 1: The study of relatively smooth spatial curves and surfaces constitutes one of the main objectives of multivariate calculus.

The present work addresses a few of the many important concepts and methods in differential calculus. The term 'mosaic' in its title reflects the fact that our exposition follows a sequence of relatively self-contained topics that can, nevertheless, be combined in the modeling and solution of several practical problems. In addition, we tried to follow a relatively logic progression of subjects.

It is often easier to say what something is not than trying to provide a definition of what it is. Let's contemplate the easiest issue first: the present work is by no mean intended to be a program in calculus. In particular, it does not include a more formal and systematic treatment of the main results, with respective proofs. Also, reflecting

its more practical perspective, we have assumed that all the considered differentials exist, not covering respective existence conditions. In addition, this text is not comprehensive: as a consequence of its limited length, several important concepts and methods (e.g. series, limits, continuity, torsion, integration and differentiation techniques, Green and other theorems, etc.) have not been included. The developments are also restricted to 2D and 3D.

Instead, the present work is mainly intended as a brief presentation of a few of the principal topics in differential calculus, which can be particularly useful for those interested to having a first contact with multivariate calculus while disposing of a little time. It may also serve as a summary refresher.

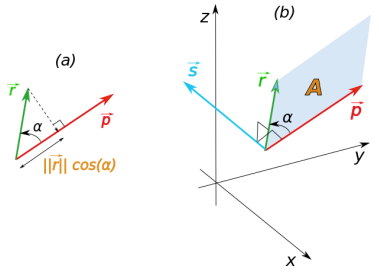

Figure 2: (a): The *inner product* between $\vec{p}$ and $\vec{r}$ corresponds to the product of $||\vec{p}||$ with the projection $||\vec{r}||cos(\alpha)$ of $\vec{r}$ on $\vec{p}$. (b): The *cross product* between two vectors $\vec{p}$ and $\vec{r}$ is a new vector $\vec{s}$ orthogonal to them and following the right-hand rule. The magnitude of $\vec{s}$ corresponds to the area A of the respectively defined parallelogram. Observe that, in this work, the (x, y, z) coordinate system also follows the right-hand rule.

2 Vector Spaces and Mutivariate Functions

Multivariate calculus involves variables defined in multi-dimensional vector spaces, such as $\Re^2$, $\Re^3$ and $\Re^N$. In this work, we focus on $\Re^3$, but $\Re^2$ can be immediately addressed as it is contained in $\Re^3$. A point $\vec{p}$ in $\Re^3$ will be henceforth expressed as:

$$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x; y; x] = (x, y, z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}, \quad (1)$$

where $\vec{\imath}$, $\vec{\imath}$ and $\vec{\imath}$ are *versors* of $\Re^3$, in the sense that $||\vec{\imath}|| = ||\vec{\jmath}|| = ||\vec{k}|| = 1$, defining a *basis* for the respective vector space $\Re^3$. Recall that $||\vec{p}|| = \sqrt{x^2 + y^2 + z^2}$.

Multivariate calculus relates closely to analytic geometry. The *inner* (or scalar) *product* between two vectors $\vec{p} = (p_x, p_y, p_z)$ and $\vec{r} = (r_x, r_y, r_z)$ can be defined as corresponding to:

$$\vec{p} \cdot \vec{r} = \vec{r} \cdot \vec{p} = \langle \vec{p}, \vec{r} \rangle = || \vec{p} \, || \, || \vec{r} \, || \, cos(\alpha), \quad (2)$$

where α is the smallest angle between $\vec{p}$ and $\vec{r}$.

As such, the inner product can be interpreted as reflecting the similarity between the inclinations of the two vectors, achieving its maximum when the two vectors are perfectly aligned (i.e. $\alpha = 0$), minimum value when they oppose one another (i.e. $\alpha = \pi$) and nullifying (*orthogonal*) when (i.e. $\alpha = \pi/2$). Figure 2(a) illustrates the scalar product between two generic vectors $\vec{p}$ and $\vec{r}$.

The *cross product* between two vectors $\vec{p} = (p_x, p_y, p_z)$ and $\vec{r} = (r_x, r_y, r_z)$ can be defined as:

$$\vec{p} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ p_x & p_y & p_z \\ r_x & r_y & r_z \end{vmatrix} = \vec{s} = || \vec{p} \, || \, || \vec{r} \, || \, sin(\alpha) \, \hat{n}, \quad (3)$$

where $\vec{v}$ is a versor orthogonal to both $\vec{p}$ and $\vec{r}$, and α is the smallest angle between the two original vectors.

The cross product results a new vector that is orthogonal to both $\vec{p}$ and $\vec{r}$ and which follows the *right-hand rule*. The magnitude of the cross product can be understood as corresponding to the area of the parallelogram defined by the two vectors, as illustrated in Figure 2(b)

An *implicit function* can be defined as $\psi(x, y, z) = 0$. For instance, the implicit equation of a plan orthogonal to a versor $\hat{r} = (r_x, r_y, r_z)$ and at distance d from the coordinate origin can be expressed by considering that the scalar product between any point (x, y, z) on the plane and the versor $\hat{r}$ is constant and equal to d, i.e.:

$$(x, y, z) \cdot (r_x, r_y, r_z) = r_x x + r_y y + r_z z = d. \quad (4)$$

Observe that, if we isolate z at the lefthand side of the equation, we get $z = -r_x/r_z \; x - r_y/r_z \; y + d$, which can be recognized as the more traditional plan equation.

A *multivariate function* (or transformation, vector function) acting from $\Re^2$ into $\Re^M$, can be understood as a set of functions:

$$\begin{cases} \psi_1 = \psi_1(x, y, z) \\ \psi_2 = \psi_2(x, y, z) \\ \dots \\ \psi_M = \psi_M(x, y, z). \end{cases}$$

Each of these functions can be though of as a *scalar field* on (x, y, z), characterized by the association of a real value to every point in the $\Re^3$ domain.

The *composition* of multivariate functions has the general form:

$$\begin{cases} \xi_1(\psi_1, \dots, \psi_M) = \psi_1(\psi_1(x, y, z), \dots, \psi_M(x, y, z)) \\ \xi_2(\psi_1, \dots, \psi_M) = \psi_1(\psi_1(x, y, z), \dots, \psi_M(x, y, z)) \\ \dots \\ \xi_Q(\psi_1, \dots, \psi_M) = \psi_1(\psi_1(x, y, z), \dots, \psi_M(x, y, z)). \end{cases} \quad (5)$$

for some integer $Q > 0$.

3 Partial Derivatives

Multivariable calculus is characterized by the extensive use of *partial derivatives*. Given a scalar field $\psi(x,y,z)$ in $\Re^3$, its possible partial derivatives are given as:

$$\frac{\partial\psi(x,y,z)}{\partial x} = \lim_{\Delta x\to 0}\frac{\psi(x+\Delta x,y,z)-\psi(x+\Delta x,y,z)}{\Delta x} \quad (6)$$

$$\frac{\partial\psi(x,y,z)}{\partial y} = \lim_{\Delta y\to 0}\frac{\psi(x,y+\Delta y,z)-\psi(x,y+\Delta y,z)}{\Delta y} \quad (7)$$

$$\frac{\partial\psi(x,y,z)}{\partial z} = \lim_{\Delta z\to 0}\frac{\psi(x,y,z+\Delta z)-\psi(x,y,z+\Delta z)}{\Delta z}. \quad (8)$$

For instance, consider the simple scalar field expressed as:

$$\psi(x,y,z) = \frac{1}{2}\left(x^2+y^2+z^2\right). \quad (9)$$

Its respective partial derivatives are:

$$\frac{\partial\psi(x,y,z)}{\partial x} = x; \quad \frac{\partial\psi(x,y,z)}{\partial y} = y; \quad \frac{\partial\psi(x,y,z)}{\partial z} = z..$$

Each of these quantities expresses the tendency of the scalar field $\psi(x,y,z)$ to vary along the respective direction, e.g. $\frac{\partial\psi(x,y,z)}{\partial x}$ quantifies the tendency of the field to grow as a consequence of an infinitesimally small displacement ∂x along the $x-$axis.

In the case of multivariate functions as in Equation 5, the following chain rule applies:

$$\frac{\partial\xi_i}{\partial x} = \frac{\partial\xi_i}{\partial\psi_1}\frac{\partial\psi_1}{\partial x} + \frac{\partial\xi_i}{\partial\psi_2}\frac{\partial\psi_2}{\partial x} + \ldots + \frac{\partial\xi_i}{\partial\psi_M}\frac{\partial\psi_M}{\partial x}. \quad (10)$$

4 Differential Operators

Partial derivatives can be combined in several ways, yielding respective *differential operators*.

For instance, the *gradient* of a scalar field $\psi(x,y,z)$ corresponds to the following *vector field*:

$$\vec{\nabla}\psi(x,y,z) = \frac{\partial\psi(x,y,z)}{\partial x}\hat{\imath} + \frac{\partial\psi(x,y,z)}{\partial y}\hat{\jmath} + \frac{\partial\psi(x,y,z)}{\partial z}\hat{k}. \quad (11)$$

Observe that the $\vec{\nabla}$ operator can be conceptualized as:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}. \quad (12)$$

The gradient of $\psi(x,y,z)$ at a specific point can be understood as the vector defined on $\Re^3$ *pointing towards, and with magnitude proportional to, the maximum positive variation* of the scalar field $\psi(x,y,z)$ at that point.

In the case of $\psi(x,y,z=x+y+z)$, we have:

$$\vec{\nabla}\psi(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = [x;y;z]. \quad (13)$$

Which corresponds to a radial vector field centered at the coordinate origin and with magnitude $||\vec{\nabla}\psi(x,y,z)|| = \sqrt{x^2+y^2+z^2}$.

Now, let's consider a *vector field* $\vec{\phi}(x,y,z) = \phi_x(x,y,z)\hat{\imath} + \phi_y(x,y,z)\hat{\imath} + \phi_z(x,y,z) + \hat{k}$, which is therefore composed of three respective scalar fields $\phi_x(x,y,z)$, $\phi_y(x,y,z)$ and $\phi_z(x,y,z)$. Its respective *divergent* can be defined as the following *scalar field*, obtained by the inner (or scalar) product of $\vec{\nabla}$ with $\vec{\phi}(x,y,z)$:

$$\vec{\nabla}\cdot\vec{\phi}(x,y,z) = \frac{\partial\phi_x(x,y,z)}{\partial x} + \frac{\partial\phi_y(x,y,z)}{\partial y} + \frac{\partial\phi_z(x,y,z)}{\partial z} \quad (14)$$

The divergent of a vector field $\vec{\nabla}\cdot\vec{\phi}(x,y,z)$ can be understood as the sum of: the variation of its component $\phi_x(x,y,z)$ along the $x-$axis; the variation of $\phi_y(x,y,z)$ along the $y-$axis, and the variation of its $\phi_z(x,y,z)$ along the $z-$axis. Therefore, the divergent of a vector field reflects the tendency of the constituent vectors to vary, along its respective domain, in magnitude and/or orientation. As a consequence, the divergent of any constant vector is necessarily equal to zero.

Informally speaking, the value of the divergent of a vector field at a given point can be conceptualized as the 'expansion' (or 'contraction' for negative values) of the field within an infinitesimal region around that point.

As an example, let's derive the divergent of the vector field $\vec{\phi}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, along the space $\Re^3$:

$$\vec{\nabla}\cdot\vec{\phi}(x,y,z) = 1+1+1 = 3. \quad (15)$$

Given a *vector field* $\vec{\phi}(x,y,z)$, it is also possible to define its respective *curl* as the *vector field* corresponding to the following cross product:

$$\vec{\nabla}\times\vec{\phi}(x,y,z) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix} =$$

$$= \left(\frac{\partial\phi_z}{\partial y} - \frac{\partial\phi_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial\phi_x}{\partial z} - \frac{\partial\phi_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial\phi_y}{\partial x} - \frac{\partial\phi_x}{\partial y}\right)\hat{k}. \quad (16)$$

The *Laplacian* of a scalar field $\psi(x,y,z)$ is a second-order operator that can be defined as:

$$\nabla^2\psi(x,y,z) = \frac{\partial^2\psi(x,y,z)}{\partial x^2} + \frac{\partial^2\psi(x,y,z)}{\partial y^2} + \frac{\partial^2\psi(x,y,z)}{\partial z^2} \quad (17)$$

5 The Jacobian Matrix of a Multiravirate Function

Given a multivariate function $S:\Re^3\to\Re^3$, such as:

$$F: \begin{cases} F_x = \psi_1(x,y,z); \\ F_y = \psi_2(x,y,z); \\ F_z = \psi_3(x,y,z). \end{cases} \quad (18)$$

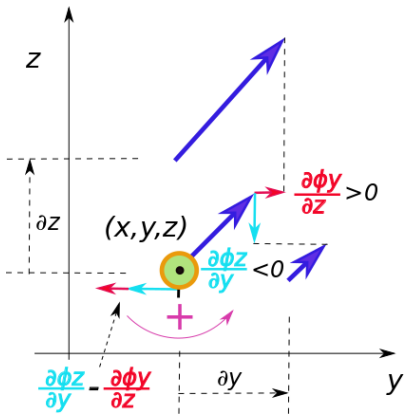

Figure 3: Interpretation of the component $\left(\frac{\partial \phi_z}{\partial y} - \frac{\partial \phi_y}{\partial z}\right)\hat{\imath}$ of the curl operator as corresponding to the difference between the variation of the vectors along the y and z directions. The curly arrow indicates that the resulting value follows the right-hand rule, implying that the negative difference value obtained in this particular illustration at the green point implies clockwise rotation.

its respective *Jacobian matrix* can be expressed as:

$$J(F) = \begin{bmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} & \frac{\partial \psi_1}{\partial z} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} & \frac{\partial \psi_2}{\partial z} \\ \frac{\partial \psi_3}{\partial x} & \frac{\partial \psi_3}{\partial y} & \frac{\partial \psi_3}{\partial z}. \end{bmatrix} \tag{19}$$

The determinant of the Jacobian matrix of a multivariate transformation at a specific point (x, y, z) corresponds to the ratio of expansion or shrinking of the infinitesimal space volume around that point implemented by this transformation.

6 First and Second Order Taylor Expansion of a Scalar Field

The first-order Taylor expansion of a scalar field $\psi(x, y, z)$, the direct counterpart of the first order Taylor series of a function in one-dimensional calculus, can be written as:

$$\psi(\vec{p} + \vec{\Delta}) \approx \psi(\vec{p}) + \vec{\nabla}\psi(\vec{p}) \cdot \vec{\Delta}^T, \tag{20}$$

where $\vec{\Delta} = \Delta x\, \hat{\imath} + \Delta y\, \hat{\imath} + \Delta z\, \hat{k} = [\Delta x; \Delta y; \Delta z]$.

The second order Taylor expasion of $\psi(x, y, z)$ is given as:

$$\psi(\vec{p} + \vec{\Delta}) \approx \psi(\vec{p}) + \vec{\nabla}\psi(\vec{p}) \cdot \vec{\Delta}^T + \frac{1}{2}\vec{\Delta} H_{\psi(\vec{p})}(\vec{p}) \vec{\Delta}^T, \tag{21}$$

where $H_{\psi(\vec{p})}$ is the Hessian of the scalar field $\psi(\vec{p})$, given as:

$$H_{\psi(\vec{p})} = \begin{bmatrix} \frac{\partial^2 \psi}{\partial^2 x} & \frac{\partial^2 \psi}{\partial x \partial y} & \frac{\partial^2 \psi}{\partial x \partial z} \\ \frac{\partial^2 \psi}{\partial y \partial x} & \frac{\partial^2 \psi}{\partial^2 y} & \frac{\partial^2 \psi}{\partial y \partial z} \\ \frac{\partial^2 \psi}{\partial z \partial x} & \frac{\partial^2 \psi}{\partial z \partial y} & \frac{\partial^2 \psi}{\partial^2 z} \end{bmatrix} \tag{22}$$

As an examples, let's consider the second order Taylor expansion of the scalar field $\psi(x, y, z) = 1/3(x^3 + y^3 + x^3)$. In this case we have $\vec{\nabla}\psi(\vec{p}) = \left[x^2; y^2; z^2\right]$ and

$$H_{\psi(\vec{p})}(x, y, z) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{23}$$

So that the respective second order expansion is given as:

$$\psi(\vec{p} + \vec{\Delta}) \approx \frac{1}{2}(x^3 + y^3 + z^3) + \\ + x^2\Delta x + y^2\Delta y + z^2\Delta z + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2. \tag{24}$$

If we make $\vec{p} = [2; 1; -2]$ and $\vec{\Delta} = [0.02; -0.03; -0.01]$, we have $\psi(\vec{p} + \vec{\Delta}) = 0.344826...$, while the respective first order expansion is equal to $0.343333...$ and the second order expansion yields, in the case of the current example, a better estimative $0.344733...$. Observe that we are approximating a cubic scalar field in terms of second-order expansion.

7 Extreme Values

In univariate calculus, it is known that the *extreme* (i.e. maximum or minimum) value of a function $f(x)$ takes place whenever $df(x)/dx = 0$, corresponding to a *local maximum* if $d^2 f(x)/dx^2 < 0$ or a *local minimum* when $d^2 f(x)/dx^2 > 0$.

The characterization of extreme values of a scalar field $\psi(x, y, z)$ in multivariate calculus is a little bit more intricate and involves the consideration of both the gradient and Hessian matrix of that field.

First, we identify the *critical points* of $\psi(x, y, z)$, which are characterized by $||\vec{\nabla}\psi(x, y, z)|| = 0$. Let $(\tilde{x}, \tilde{y}, \tilde{z})$ be a critical point. Every local maximum or minimum of $\psi(x, y, z)$ is a critical point, but not vice-versa. The following results can be considered while classifying a critical point $(\tilde{x}, \tilde{y}, \tilde{z})$:

1. **Positive-definite Hessian:** All eigenvalues of $H(\tilde{x}, \tilde{y}, \tilde{z})$ are *positive* $\implies$ $(\tilde{x}, \tilde{y}, \tilde{z})$ is a *local minimum point*;
2. **Negative-definite Hessian:** All eigenvalues of $H(\tilde{x}, \tilde{y}, \tilde{z})$ are *negative* $\implies$ $(\tilde{x}, \tilde{y}, \tilde{z})$ is a *local maximum point*;
3. **Indefinite Hessian:** The eigenvalues of $H(\tilde{x}, \tilde{y}, \tilde{z})$ are a mixture of *positive and negative* values $\implies$ $(\tilde{x}, \tilde{y}, \tilde{z})$ is a *saddle point*;
4. **Otherwise:** One or more null eigenvalues $\implies$ additional analysis is needed.

For instance, consider the scalar field $\psi(x, y, z) = 1/2\left[x^2 + y^2 + z^2\right]$ (a paraboloid). This field has

$\vec{\nabla}\psi(x,y,z) = [x;y;z]$, which becomes null at $(0,0,0)$, corresponding to the only critical point of $\psi(x,y,z)$. The Hessian of this field has already been calculated as in Equation 23 and corresponds to a diagonal matrix with all elements identical to 2, which implies three identical positive eigenvalues 2. This result indicates that the critical point $(0,0,0)$ is a local minimum of this scalar field, which indeed it is.

8 Parametric Curves

Parametric curves are curves in $\Re^N$ that are defined on a 1D domain parameter. Figure 4 illustrates a parametric curve in $\Re^2$.

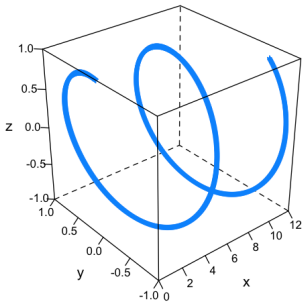

Figure 4: The helix defined by the parametric curve $\vec{\Gamma}(t) = [t; sin(t); cos(t)]$ for $t \in [0, 4\pi]$.

A *parametric curve* in 3D can be defined as:

$$\vec{\Gamma} : t \in [a,b] \longrightarrow \vec{\Gamma}(t) = [x(t); y(t); z(t)] \in \Re^3 \qquad (25)$$

where t is the *parameter* and $a, b \in \Re$. Several types of parameters – e.g. time, angle, arc length – are possible. In case the derivatives $\dot{x}(t) = dx(t)/dt$, $\dot{y}(t) = dy(t)/dt$ and $\dot{z}(t) = dz(t)/dt$ exist, the curve is said to be *differentiable*. If, in addition, these derivatives never simultaneously take zero value, the curve is said to be *regular* and, it the three derivatives are also continuous, we have a *smooth* curve.

Given a parametric curve, we can be interested in several of its properties, such as its derivatives, tangent and normal fields, arc-length, curvature, etc. (e.g. [8]). In this section we will discuss some of these quantities.

The arc-length $s(\vec{\Gamma}(t))$ of a curve between the parameter values t_1 ad t_2 can be immediately expressed as:

$$s_{\vec{\Gamma}}(t) = \int_{t_1}^{t_2} \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2} dt \qquad (26)$$

As an example, let's calculate the arc-length of the parametric curve:

$$\vec{\Gamma}(t) = [t; sin(t); cos(t)] \qquad (27)$$

shown in Figure 4 for $t_1 = 0$ and $t_2 = t$, which yields:

$$s_{\vec{\Gamma}}(t) = \int_0^t \sqrt{1 + (cos(\theta))^2 + (-sin(\theta))^2} d\theta = \\ = \left[\sqrt{2}\theta\right]\Big|_0^t = \sqrt{2}t \qquad (28)$$

In case $\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t) = 1$, we necessarily have that the parameter t of the respective curve Γ corresponds to the arc-length. In other words, the curve is said to be in *arc-length* parametrization.

The *tangent vector* (or *velocity vector*) of a parametric curve $\vec{\Gamma}$ is given as:

$$\vec{T}_{\vec{\Gamma}}(t) = \frac{d\vec{\Gamma}}{dt} = \dot{\vec{\Gamma}}(t) = [\dot{x}(t); \dot{y}(t); \dot{z}(t)] \qquad (29)$$

The *normalized tangent vector* of that same curve, now having unit magnitude, can be expressed as:

$$\vec{\tau}_{\vec{\Gamma}}(t) = \frac{[\dot{x}(t); \dot{y}(t); \dot{z}(t)]}{||\,[\dot{x}(t); \dot{y}(t); \dot{z}(t)]\,||} \qquad (30)$$

Let's calculate the tangent vector for the curve in Equation 27:

$$\vec{T}_{\vec{\Gamma}}(t) = [1; cos(t); -sin(t)]\,, \qquad (31)$$

and now its normalized tangent vector:

$$\vec{\tau}_{\vec{\Gamma}}(t) = \frac{1}{\sqrt{2}}[1; cos(t); -sin(t)]\,. \qquad (32)$$

As for $\vec{\Gamma}(t)$ we have that $(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2 \neq 1$, it follows that this curve is not in arc-length parametrization.

In case the tangent vector of a curve $\vec{\Gamma}$ is not equal to $[0;0;0]$ whenever the considered values of t, $\vec{\Gamma}$ is said to be *regular*. A value of t for which the tangent vector is nullified is called a *singular point*.

It is frequently preferable to work with a curve in its arc-length parametrization. Otherwise, the curve can be reparametrized to arc-length as follows. First, find an expression for the arc-length of the curve by using Equation 26, which will yield a respective expression $s = s(t)$. If possible, obtain the inverse $t = t(s)$ and substitute it in the original curve equation, which will then become arc-length parametrized.

Let's consider the reparametrization of the helix in Equation 27. We already found that its arc-length is $s(t) = \sqrt{2}t$, which yields $t = \sqrt{2}/2s$, so that we get the new curve $\tilde{\vec{\Gamma}}(s) = \left[\sqrt{2}/2s, sin(\sqrt{2}/2s); cos(\sqrt{2}/2s)\right]$, which is now arc-length parametrized.

Given a curve $\vec{\Gamma}(s)$ in arc-length parametrization, its respective *curvature* can be defined as:

$$k(s) = \sqrt{(\ddot{x}(s))^2 + (\ddot{y}(s))^2 + (\ddot{z}(s))^2} \qquad (33)$$

For the arc-length parametrized curve $\tilde{\Gamma}(s) = \left[\sqrt{2}/2s, sin(\sqrt{2}/2s); cos(\sqrt{2}/2s)\right]$ we have that:

$$\dot{\vec{\Gamma}}(s) = \left[\sqrt{2}/2, \sqrt{2}/2cos(\sqrt{2}/2s); -\sqrt{2}/2sin(\sqrt{2}/2s)\right]$$
$$\text{and } \ddot{\vec{\Gamma}}(s) = \left[0, -1/2cos(\sqrt{2}/2s); -1/2sin(\sqrt{2}/2s)\right] \Rightarrow$$
$$k(s) = \sqrt{0 + 1/2(-sin(\sqrt{2}/2s))^2 + 1/2(-cos(\sqrt{2}/2))^2} = 1.$$

We also have that a straight line will have null curvature at each of its point, and that the curvature of any point in a circle of radius r will be $k = 1/r$.

9 Parametric Surfaces

Similarly to parametric curves, it is possible to define *parametric surfaces* in $\Re^3$ in terms of a lower dimensional domain $(X, Y) \in \Re^2$ mapped into vector field, i.e.:

$$S(x, y, z) : (x(X, Y), y(X, Y), z(X, Y)), \qquad (34)$$

where X and Y are the *parameters*.

While surfaces can be defined implicitly (see Section 2), or as a vector field of the type $(x, y, f(x, y))$, parametric curves allow enhanced versatility while defining several types of surfaces, and also helps in variables or coordinate frame transformations. For instance, there are several types of surfaces that cannot be defined in the form $(x, y, f(x, y))$ but can be conveniently represented in parametric manner.

Let's consider that we want to define a parametric plan at distance d from the coordinates origin and being normal to a versor $\hat{r}$. This can be obtained by defining two versors $\hat{p}$ and $\hat{q}$ that are orthogonal one another and both orthogonal to $\hat{r}$. The points along the plan surface can then be expressed as linear combinations of $\hat{p}$ and $\hat{q}$, i.e. $[x; y; z] = X\hat{p} + Y\hat{q}$. Observe that this effectively implements a parametric plan $(x(X, Y), y(X, Y), z(X, Y))$.

Let's consider an example where $\hat{r} = c(\hat{\imath} + \hat{\jmath} + \hat{k})$, with $c = \sqrt{3}/3$. The implicit equation of this plan (see Section 2) is given as $x + y + z = d/c$.

This plan crosses the coordinate plan $(x, y, 0)$ along the line $x + y = d/c$. Let's derive $\hat{p}$ from the vector $\vec{p}$ as extending from the point defined by the crossing of the latter line with the $y-$axis, i.e. $(0, d/c, 0)$, to the point defined by the crossing of this same line with the $x-$axis, i.e. $(d/c, 0, 0)$, yielding $\vec{p} = (d/c, 0, 0) - (0, d/c, 0) = (d/c, -d/c, 0)$. We can adopt the respective versor $\hat{p} = \vec{p}/||\vec{p}|| = e(1, -1, 0)$, where $e = \sqrt{2}/2$. Thus, for $\hat{p} = e(\hat{\imath} - \hat{\jmath})$, we can obtain a possible vector $\hat{q}$ so that $\hat{p} \times \hat{q} = \hat{r}$ (a right-hand oriented relationship) as:

$$\hat{q} = \hat{r} \times \hat{p} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ c & c & c \\ e & -e & 0 \end{vmatrix} = ce\ \hat{\imath} + ce\hat{\jmath} - 2ce\ \hat{k}. \qquad (35)$$

The sought plan can now be expressed as:

$$(x, y, z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = X\hat{p} + Y\hat{q} + d\,\hat{r} =$$
$$= eX\hat{\imath} - eX\hat{\jmath} + ceY\hat{\imath} + ceY\hat{\jmath} - 2ceY\hat{k} + cd(\hat{\imath} + \hat{\jmath} + \hat{k}) =$$
$$= (ceY + eX + cd)\hat{\imath} + (ceY - eX + cd)\hat{\jmath} + (cd - 2ceY)\hat{k},$$

from which we obtain:

$$S(x, y, z) : \begin{cases} x(X, Y) = eX + ceY + cd \\ y(X, Y) = -eX + ceY + cd \\ z(X, Y) = -2ceY + cd \end{cases}$$

As another example of parametric surface, the thorus with inner radius r and outer radius R, centered at $(0, 0, 0)$ can be expressed as follows:

$$(x(u, v), y(u, v), z(u, v)) : \begin{cases} x(u, v) = rsin(v) \\ y(u, v) = [R + rcos(v)]sin(u) \\ z(u, v) = [R + rcos(v)]cos(u) \end{cases}$$

Going back to the above derived plan, defined by $\hat{r}$ and d, we observe that, though we could express the original coordinates (x, y, z) in terms of the parameters (X, Y), it is impossible to obtain an inverse relationship — i.e. express (X, Y) in terms of (x, y, z) — because a generic point (x, y, z) may not be contained in the plan expressed in terms of (X, Y).

However, in this case, we can define a new variable Z as corresponding to the coordinate along the versor $\hat{r} = \hat{p} \times \hat{q}$, therefore defining a right-hand oriented new coordinate system $(X, Y, Z) = X\hat{p} + Y\hat{q} + Z\hat{r}$ which, similarly to the coordinate system (x, y, z), is also capable of spanning $\Re^3$.

Let's now express (x, y, z) in terms of (X, Y, Z). Recall that $\hat{p} = e(\hat{\imath} - \hat{\jmath})$, $\hat{q} = ce(\hat{\imath} + \hat{\jmath} - 2\,\hat{k})$, and $\hat{r} = c(\hat{\imath} + \hat{\jmath} + \hat{k})$, and make:

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} = X\hat{p} + Y\hat{q} + Z\hat{r} + d\hat{r} =$$
$$= eX(\hat{\imath} - \hat{\jmath}) + ceY(\hat{\imath} + \hat{\jmath} - 2\hat{k}) + c(Z + d)(\hat{\imath} + \hat{\jmath} + \hat{k}) =$$
$$= (eX + ceY + c(Z + d))\hat{\imath} + (-eX + ceY + c(Z + d))\hat{\jmath} +$$
$$(-2ceY + c(Z + d))\hat{k},$$

from which we obtain the *coordinate transformation* $(x(X, Y, Z), y(X, Y, Z), z(X, Y, Z))$:

$$\begin{cases} x(X, Y, Z) = eX + ceY + cZ + cd \\ y(X, Y, Z) = -eX + ceY + cZ + cd \\ z(X, Y, Z) = -2ceY + cZ + cd, \end{cases}$$

or, in matrix form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e & ce & c \\ -e & ce & c \\ 0 & -2ce & c \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} cd \\ cd \\ cd \end{bmatrix}.$$

It is also possible to express of (X, Y, Z) in terms of (x, y, z):

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} e & ce & c \\ -e & ce & c \\ 0 & -2ce & c \end{bmatrix}^{-1} \begin{bmatrix} x - cd \\ y - cd \\ z - cd \end{bmatrix} = \\ = \begin{bmatrix} \frac{1}{2e} & -\frac{1}{2e} & 0 \\ \frac{1}{6ce} & \frac{1}{6ce} & -\frac{1}{3ce} \\ \frac{1}{3c} & \frac{1}{3c} & \frac{1}{3c} \end{bmatrix} \begin{bmatrix} x - cd \\ y - cd \\ z - cd \end{bmatrix}. \quad (36)$$

The reader is encouraged to generalize the above development for the case of having a generic versor $\hat{r} = (\alpha, \beta, \gamma)/\sqrt{\alpha^2 + \beta^2 + \gamma^2}$.

More generic parametric surfaces can be defined. For instance, we can have $(X, Y, X^2 + Y^2)$.

which can then be obtained in the (x, y, z) coordinates as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e & ce & c \\ -e & ce & c \\ 0 & -2ce & c \end{bmatrix} \begin{bmatrix} X \\ Y \\ X^2 + Y^2 \end{bmatrix} + \begin{bmatrix} cd \\ cd \\ cd \end{bmatrix}.$$

Figure 5 illustrates the above parabolic parametric function considering $d = 1$.

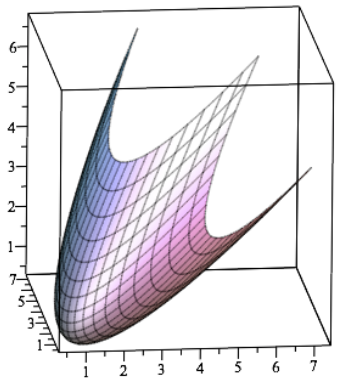

Figure 5: The parabolic parametric surface in the discussed example considering $d = 1$.

10 Local Frames of Parametric Surfaces

Let $(\chi_x(u, v), \chi_y(u, v), \chi_z(u, v))$ be a parametric surface on $\Re^3$, which can be represented vectorially as:

$$\vec{p}(u, x) = \chi_x(u, v)\hat{\imath} + \chi_y(u, v)\hat{\jmath} + \chi_z(u, v)\hat{k}, \quad (37)$$

where $\vec{p}$ is a vector that spans the surface χ.

It is interesting to consider the respective partial derivatives:

$$\frac{\partial \vec{p}}{\partial u} = \frac{\partial \chi_x(u, v)}{\partial u}\hat{\imath} + \frac{\partial \chi_y(u, v)}{\partial u}\hat{\jmath} + \frac{\partial \chi_z(u, v)}{\partial u}\hat{k}$$
$$\frac{\partial \vec{p}}{\partial v} = \frac{\partial \chi_x(u, v)}{\partial v}\hat{\imath} + \frac{\partial \chi_y(u, v)}{\partial v}\hat{\jmath} + \frac{\partial \chi_z(u, v)}{\partial v}\hat{k},$$

as they can be understood as providing a differential local tangent plan on each of the points $(\chi_x(u, v), \chi_y(u, v), \chi_z(u, v))$ on the surface χ.

Interestingly, the magnitude $\left|\left|\frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v}\right|\right|$ reflects the area (see Section 2) of the infinitesimal reference frame defined by $\frac{\partial \vec{p}}{\partial u}$ and $\frac{\partial \vec{p}}{\partial v}$, giving an idea of the local area of the surface around a very small neighborhood of each of its points.

The cross product $\frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v}$ defines a third differential $\frac{\partial \vec{p}}{\partial n}$ that is *normal* $(\hat{n})$ to the surface χ at each of its points:

$$d\vec{S} = \frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v}, \quad (38)$$

which is called *surface element*.

These three vectors can then be understood as a local reference frame around each point on the plan:

$$\left(\frac{\partial \vec{p}}{\partial u}, \frac{\partial \vec{p}}{\partial v}, d\vec{S}\right). \quad (39)$$

11 Surface Integrals of Scalar Fields

Given a parametric surface $\chi : (\chi_x(u, v), \chi_y(u, v), \chi_z(u, v))$ and a scalar field $\psi(x, y, z)$, the surface integral of this scalar field constrained to the parametric surface χ can be expressed as:

$$\iint_S \psi(x, y, z) dS = \\ = \iint_R \psi(\chi_x(u, v), \chi_y(u, v), \chi_z(u, v)) \left|\left|\frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v}\right|\right| du dv, \quad (40)$$

where R is the region along the 2D domain (u, v) to be considered for the integration.

Observe that the the magnitude $\left|\left|\frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v}\right|\right|$ implements a kind of 'weighting' of the scalar field ψ integrated at each point (x, y, z).

Let the scalar field $\psi(x, y, z) = x + y - z$ and the plan χ given as $x + y + z = 1$, which can be expressed parametrically as $\chi : (\chi_x(u, v) = u, \chi_y(u, v) = v, \chi_z(u, v) = 1 - u - v)$, and respective vector form $\vec{p} = (u, v, -u - v + 1)$. We have that:

$$\frac{\partial \chi_x(u, v)}{\partial u} = 1; \quad \frac{\partial \chi_x(u, v)}{\partial v} = 0 \quad (41)$$
$$\frac{\partial \chi_y(u, v)}{\partial u} = 0 \quad \frac{\partial \chi_y(u, v)}{\partial v} = 1 \quad (42)$$
$$\frac{\partial \chi_z(u, v)}{\partial u} = -1 \quad \frac{\partial \chi_z(u, v)}{\partial v} = -1, \quad (43)$$

from which we can obtain:

$$\frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} + \hat{k}. \tag{44}$$

It follows that:

$$\left|\left| \frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v} \right|\right| = \sqrt{3}. \tag{45}$$

Observe that the obtained constant value is compatible with the fact that we are integrating with respect to a plan, which would indeed be expected to imply the same 'weighting' on its entire surface.

We are now ready to calculate the integral of the scalar field $\psi(x, y, z)$ restricted to the parametric surface χ for the region R defined by the intervals $0 \leq u \leq 1$ and $0 \leq v \leq 1$ as:

$$\begin{aligned}
&\iint_R \psi(\chi_x(u,v), \chi_y(u,v), \chi_z(u,v)) \left|\left| \frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v} \right|\right| dudv = \\
&= \sqrt{3} \int_0^1 \int_0^1 (\psi_x(u,v) + \psi_y(u,v) - \psi_z(u,v)) dudv = \\
&= \sqrt{3} \int_0^1 \int_0^1 (u + v - (1 - u - v)) dudv = \\
&= \sqrt{3} \int_0^1 \int_0^1 (2u + 2v - 1) dudv = \\
&= \sqrt{3} \int_0^1 \left(u^2 + 2uv - u\right) \Big|_0^1 dv = \\
&= \sqrt{3} \int_0^1 (1 + 2v - 1)\, dv = \\
&= \sqrt{3} \left(v + v^2 - v\right) \Big|_0^1 = \sqrt{3}.
\end{aligned}$$

Note that $\psi(\chi_x(u,v), \chi_y(u,v), \chi_z(u,v))$ corresponds to the slicing of the scalar field ψ by the considered parametric surface χ which is, in the specific case of this example, a plan.

12 Surface Integrals of Vector Fields

Similarly to integrating a scalar field on a given parametric surface, it is also possible to integrate a *vector field* along a generic parametric surface $\chi(\chi_x(u,v), \chi_y(u,v), \chi_z(u,v))$ as:

$$\begin{aligned}
&\iint_S \vec{\phi}(x,y,z) \cdot d\vec{S} = \iint_S \left(\vec{\phi} \cdot \vec{n}\right) dS = \\
&= \iint_R \chi(\psi_x(u,v), \chi_y(u,v), \chi_z(u,v)) \cdot \left(\frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v}\right) dudv.
\end{aligned} \tag{46}$$

As an example, let's consider the same parametric surface χ as in the example in the previous seciton, and that the vector field to be integrated on this surface is equal to $\vec{\phi} = (x, -y, z)$. We have that

$$\begin{aligned}
&\iint_R \psi(\chi_x(u,v), \chi_y(u,v), \chi_z(u,v)) \cdot \left(\frac{\partial \vec{p}}{\partial u} \times \frac{\partial \vec{p}}{\partial v}\right) dudv = \\
&= \int_0^1 \int_0^1 (u, -v, 1 - u - v) \cdot (1, 1, 1) dudv = \\
&= \int_0^1 \int_0^1 (u - v + 1 - u - v) dudv = \\
&= \int_0^1 \int_0^1 (1 - 2v) dudv = \int_0^1 (1 - 2v) dv = \\
&= \int_0^1 (v - v^2) dv = (v - v^2) \Big|_0^1 = 0.
\end{aligned} \tag{47}$$

which is expected as a consequence of the relative symmetry between the reference parametric surface and the assumed vector field.

13 Cylindrical and Spherical Coordinates

It is often interesting, or necessary, to use alternative coordinate systems other than the rectangular coordinates that we have considered so far in this work. Two such systems are often applied: *cylindrical coordinates* and *spherical coordinates*.

The cylindrical coordinates system stems from the mapping:

$$x = \rho cos\theta; \quad y = \rho sin\theta; \quad z = z.$$

It can be shown (e.g. [9]) that in this system, we have:

$$\begin{aligned}
\vec{\nabla}\psi(\rho,\theta,z) &= \frac{\partial \psi_\rho}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{\partial \psi}{\partial z}\hat{e}_z \\
\vec{\nabla} \cdot \vec{\phi}(\rho,\theta,z) &= \frac{1}{\rho}\frac{\rho \partial \phi_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial \phi_\theta}{\partial \theta} + \frac{\partial \phi_z}{\partial z} \\
\vec{\nabla} \times \vec{\phi}(\rho,\theta,z) &= \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \phi_\rho & \rho\phi_\theta & \phi_z \end{vmatrix}.
\end{aligned}$$

The spherical coordinate system is obtained by making:

$$x = \rho cos(\beta)sin(\theta);\ y = \rho sin(\beta)sin(\theta);\ z = \rho cos(\theta), \tag{48}$$

implying:

$$\vec{\nabla}\psi(\rho,\theta,z) = \frac{\partial\psi_\rho}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial\psi}{\partial\theta}\hat{e}_\theta + \frac{1}{\rho sin(\theta)}\frac{\partial\psi}{\partial\beta}\hat{e}_\beta$$

$$\vec{\nabla}\cdot\vec{\phi}(\rho,\theta,z) =$$

$$= \frac{1}{\rho^2}\frac{\partial(\rho^2\phi_\rho)}{\partial\rho} + \frac{1}{\rho sin(\theta)}\frac{\partial(\phi_\theta sin(\theta))}{\partial\theta} + \frac{1}{rsin(\theta)}\frac{\partial\phi_\beta}{\partial\beta}$$

$$\vec{\nabla}\times\vec{\phi}(\rho,\theta,z) = \begin{vmatrix} \frac{1}{\rho^2 sin(\theta)}\hat{e}_\rho & \frac{1}{\rho sin(\theta)}\hat{e}_\theta & \frac{1}{\rho}\hat{e}_\beta \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\beta} \\ \phi_\rho & \rho\phi_\theta & \rho\phi_\beta sin(\theta) \end{vmatrix}.$$

14 Concluding Remarks

A few concepts and methods from Multivariate integro-differential calculus have been briefly addressed, and sometimes illustrated. Though we have not incorporated proofs or a more formal treatment, it is expected that the above presentation can help the reader to understand or revise some important topics in multivariate calculus.

Those interested to complement this presentation, or probe further into the captivating world of multivariate calculus are encouraged to refer to the respective literature (e.g. [1, 2, 3, 4]).

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